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Energy Method for Selection of Degrees of Freedom in Condensation

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An analytical method is presented for the selection of degrees of freedom in condensation of eigenproblems. The method is based on the energies associated with the degrees of freedom in the eigenmodes of structural systems. For the energy estimation Ritz vectors are calculated, using the stiffness and mass matrices. The energies added through the modes or the weighted row sum of the energy distribution matrix can be used as an effective guideline on which degrees of freedom should be retained in the analysis. Another approach of sequential selection can be employed, in which a finite number of new degrees of freedom with the largest energy are taken in each mode and the final union becomes the analysis set. The energy of the selected degrees of freedom or the column sum of the matrix can be used to predict the solution accuracy in each mode. The error analysis shows that the perturbation in eigenvalue is related with the energy of the degrees of freedom. The row and column sums indicate the completeness of the selected set and give a clue to how many degrees of freedom to be included. The energy criterion has shown that the conventional practice of choosing translational degrees of freedom is appropriate only for the lowest modes. Numerical investigations were performed to test the convergence criterion. The energy method proved to work well for typical example problems.

Introduction

FOR large structural systems only the lowest eigenmodes are calculated because of computational difficulty. Various reduction methods[1–9] are used to transform the original eigensystem into a subspace of reduced dimension. The numerical efficiency is improved at the expense of solution accuracy. For a general discussion of reduction methods for dynamic analysis, the reader is referred to the review of Ref. 10.

In condensation methods a set of degrees of freedom called the primary (master) is retained in the analysis while the secondary (slave) ones are condensed out. The primary set should describe the lowest eigenmodes accurately. A difficult problem is which and how many degrees of freedom should be included in the analysis.[11–13]

A sequential elimination of the secondary degrees of freedom is based on the highest ratio of diagonal terms of the structural matrices (k_{ii}/m_{ii}). If a reasonably good selection is made, it is known that the lowest one-third of the eigenvalues of the condensed system will be within the engineering accuracy of 5% (Ref. 7). The present study seeks to establish effective criteria for which and how many degrees of freedom should be selected.

A general principle in system condensation is that the energy should be preserved in each eigenmode. Hence, the reduced stiffness and mass matrices conserve the strain and kinetic energies. Nevertheless, the equation of motion in the reduced subspace does not satisfy the equilibrium of the original system.[14] The incorrect energy distribution over the degrees of freedom is the major source of error in the condensation. The minimization of the energy unbalance has been successfully applied to the structural optimization of dynamic systems.[15]

To reduce the energy error, one defines the primary set to take the degrees of freedom dominating the energy distribution. Because there are no a priori exact eigenmodes available, Ritz vectors[16,17] are calculated for the energy estimation. Usually, the vectors tend to contain higher modes and the relevant degrees of freedom.

The energy distribution varies from mode to mode and so does the primary set. Sometimes, the degrees of freedom with large energy in lower modes can carry small energy in intermediate modes. To take into consideration the degrees of freedom related with the accurate lowest modes, two approaches can be considered: sequential and row sum selections.

In the sequential selection a finite number of degrees of freedom with the largest energy are taken in each mode, and the final union is used in the analysis. Another simple method is to add the components of the energy distribution matrix through the modes of interest getting the row sum of energy. The degrees of freedom with the largest sum are selected for the primary set. To put more emphasis on the lowest modes, a weighted row sum can be recommended.

With mass normalization of the mode shapes, not only the energy of the degrees of freedom in each eigenmode (column sum) but also the energy of each degree of freedom for the whole eigenmodes (row sum) become unity. Hence, the partial column sum shows the contribution of the selected set in each mode and can be used to predict the solution accuracy.[18–20] The partial row sum of a degree of freedom indicates the degree of importance to the modes of interest and can be used as a selection criterion.

There have been a few research studies on the application of the energy associated with the degrees of freedom to various fields of engineering. For the optimal sensor placement[21,22] the criterion of linear independence of the primary degrees of freedom has been employed. It is believed that the procedure to be developed here is more rational than other selection methods because the physical meaning of the energy distribution is considered. The energy method can provide many applications to the cases where a reduced subsystem should represent the properties of the whole structure.

The method has proved that the usual rule of choosing translational degrees of freedom is appropriate only for the lowest modes. In the intermediate modes rotational ones gain significant energy and must be selected to get more accurate solutions. Especially in plate problems, MSC/NASTRAN[7] suggests that all of the active rotations as well as translations at a few nodes be included in the analysis.

Typical numerical examples show that the energy criteria can offer good guidelines on the application of condensation methods. MSC/NASTRAN was employed for the analysis of structural systems. The matrix manipulations with mass matrix and mode shapes were carried out through the use of the direct matrix abstraction program (DMAP).

Condensation of Eigenproblem

In finite element analysis a general eigenproblem is obtained for undamped free vibration:

$$[k]\{\phi\} = \lambda[m]\{\phi\} \tag{1}$$

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where $[k]$ and $[m]$ are the stiffness and mass matrices. An eigenpair is denoted as λ and $\{\phi\}$.

Equation (1) can be partitioned as

$$\begin{pmatrix} k_{pp} & k_{ps} \\ k_{sp} & k_{ss} \end{pmatrix} \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} = \lambda \begin{pmatrix} m_{pp} & m_{ps} \\ m_{sp} & m_{ss} \end{pmatrix} \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} \tag{2}$$

The primary degrees of freedom $\{\phi_p\}$ are included in the analysis while the secondary set $\{\phi_s\}$ is condensed out.

If the mass associated with the secondary set is neglected, the transformation is written as

$$\{\phi_s\} \cong -[k_{ss}]^{-1}[k_{sp}]\{\phi_p\} \equiv [T]\{\phi_p\} \tag{3}$$

Through the transformation an eigenproblem in the reduced subspace is obtained in terms of the primary degrees of freedom:

$$[K_r]\{\phi_p\} = \lambda_r[M_r]\{\phi_p\} \tag{4}$$

where the reduced structural matrices are written as

$$[K_r] = [k_{pp}] - [k_{ps}][k_{ss}]^{-1}[k_{sp}]$$

$$[M_r] = [m_{pp}] - [m_{ps}][k_{ss}]^{-1}[k_{sp}] - [k_{ps}][k_{ss}]^{-1}[m_{sp}] - [k_{ps}][k_{ss}]^{-1}[m_{ss}][k_{ss}]^{-1}[k_{sp}] \tag{5}$$

Energy for Modes and Degrees of Freedom

In the system condensation the reduced stiffness and mass matrices guarantee the conservation of the strain and kinetic energies. However, the solution in the reduced subspace does not satisfy the equilibrium of the original system because the distribution of energy over the primary degrees of freedom is not correct.

To introduce the energy criterion developed here, the energies associated with the degrees of freedom will be examined in greater detail. The eigenproblem of Eq. (1) has N eigenmodes and can be written in compact form as

$$[K][\Phi] = [M][\Phi][\Lambda] \tag{6}$$

where $[\Phi]$ is the modal matrix containing the mass-normalized mode shapes and $[\Lambda]$ is a diagonal matrix for the eigenvalues. The equation states that each degree of freedom should be in equilibrium of elastic and inertia forces in the deflection modes.

Premultiplying Eq. (6) by $[\Phi]^T$ gives

$$[\Phi]^T[k][\Phi] = [\Phi]^T[m][\Phi][\Lambda]$$

or

$$[K] = [M][\Lambda] \tag{7}$$

The generalized stiffness and mass matrices $[K]$ and $[M]$ are diagonal.

The equation shows not only the orthogonality of the eigenmodes but also the equilibration of the strain and kinetic energies in each mode. If the matrix $[\Phi]$ has N vectors, Eq. (7) is equivalent to Eq. (6). A set of $L(<N)$ vectors satisfying the orthogonality may not be the eigenvectors of the structure,[14] or Eq. (7) is only a necessary condition for the vectors.

A different interpretation can be explored for the generalized structural matrices. For convenience, a notation for cross multiplication of matrices is introduced:

$$[C] = [A] \otimes [B] \quad \text{or} \quad c_{ij} = a_{ij}b_{ij} \tag{8}$$

From Eq. (6) one obtains

$$[\Phi] \otimes [k][\Phi] = [\Phi] \otimes [m][\Phi][\Lambda] \tag{9}$$

which states that each degree of freedom should be in equilibrium of the strain and kinetic energies in each eigenmode. Equation (9) shows the equilibrium in terms of energy and is exactly equivalent to Eq. (6) even when $[\Phi]$ has $L(<N)$ vectors. The dynamic equilibrium should be expressed for the degrees of freedom in terms of either force or energy.

For the structural optimization of dynamic systems, the energy error in each degree of freedom was minimized to achieve the desired goal.[15] The energy approach can also be used to overcome the numerical difficulty of unit inconsistency in the convergence criteria for the nonlinear structural analysis.

For simplicity, the generalized mass matrix will be considered further:

$$[G] = [\Phi] \otimes [m][\Phi] \quad \text{or} \quad g_{ij} = \sum_{k=1}^{N} \phi_{ij} m_{ik} \phi_{kj} \tag{10}$$

The energy distribution matrix $[G]$ represents the relative magnitude of energy associated with the degrees of freedom.

Let us define the column sum of the matrix as a sum of components in a column. As is well known, the column sum of matrix $[G]$ is the total energy of the degrees of freedom in the corresponding mode.

jth diagonal term of $[\Phi]^T[m][\Phi] = [I]$:

$$\sum_{i=1}^{N} g_{ij} = \sum_{i=1}^{N}\sum_{k=1}^{N} \phi_{ij} m_{ik} \phi_{kj} = 1, \qquad j = 1, \ldots, N \tag{11}$$

Kammer[21] suggested that a partial sum of the energy ranging from 0.4 to 0.5 would be sufficient for the primary set to describe an eigenmode accurately:

$$Ec_j = \sum_{i} g_{ij} = \sum_{i}\sum_{k=1}^{N} \phi_{ij} m_{ik} \phi_{kj} \tag{12}$$

In the same way a row sum can be obtained as follows.

ith diagonal term of $[m][\Phi][\Phi]^T = [I]$:

$$\sum_{j=1}^{N} g_{ij} = \sum_{j=1}^{N}\sum_{k=1}^{N} \phi_{ij} m_{ik} \phi_{kj} = 1, \qquad i = 1, \ldots, N \tag{13}$$

The row sum of each degree of freedom becomes unity, representing the same degree of importance to the whole eigensystem.

For the lowest L eigenmodes the partial row sum is written as

$$Er_i = \sum_{j=1}^{L} g_{ij} = \sum_{j=1}^{L}\sum_{k=1}^{N} \phi_{ij} m_{ik} \phi_{kj} \tag{14}$$

Both the column and the row sums have unit value when all degrees of freedom and eigenmodes are considered. Hence, the partial column sum represents the contribution of the selected degrees of freedom in each mode and can be used to estimate the accuracy for the modes. The partial row sum of a degree of freedom indicates the degree of importance to the modes of interest and can be used as a selection criterion.

Sometimes, the nodal force is in the opposite direction of the nodal displacement, resulting in negative energy. If one considers the negative contribution as much as the positive one, the absolute sum may have a practical meaning. In practice, however, the difference is usually small in lower modes.

Ritz Vectors

Because there are no a priori eigenmodes available, the Ritz vectors are calculated for the estimation of the energy distribution. The diagonal terms of the mass matrix are used to get the first Ritz vector, which is consistent with the general practice of selection in the condensation:

$$[k]\{X^{(1)}\} = \{m_{ii}\} \tag{15}$$

Additional Ritz vectors are obtained through inverse iteration and orthonormalization with respect to the mass matrix:

$$[k]\{X^{(k+1)}\} = [m]\{X^{(k)}\} \tag{16}$$

Error in Eigenvalues

The error in eigenvalues of the reduced eigenproblem can be estimated using perturbation methods. Thomas[18] proposed a formulation as

$$|-\Delta\lambda_r/\lambda_r| \le \lambda_r/\lambda_s \tag{17}$$

where λ_s is the smallest eigenvalue in the subspace for the secondary degrees of freedom. It can be simply approximated by the lowest ratio of the diagonal terms k_{ss}/m_{ss}.

Flax[3] and Kim[19] showed a first-order perturbation equation for the eigenvalue error in condensation:

$$-\Delta\lambda_r/\lambda_r \cong \lambda_r\{f_s\}^T([k_{ss}] - \lambda_r[m_{ss}])^{-1}\{f_s\} \tag{18}$$

where

$$\{f_s\} = \left([m_{sp}] - [m_{ss}][k_{ss}]^{-1}[k_{sp}]\right)\{\phi_p\} \cong [m_{sp}]\{\phi_p\} + [m_{ss}]\{\phi_s\} \tag{19}$$

The vector $\{f_s\}$ is related with the inertia force of the secondary degrees of freedom.

The eigenproblem in the secondary subspace can be written as

$$[k_{ss}][\Psi_s] = [m_{ss}][\Psi_s][\Lambda_s] \tag{20}$$

where $[\Psi_s]$ and $[\Lambda_s]$ are the mass-normalized modal matrix and a diagonal matrix for the corresponding eigenvalues.

Using Eq. (20), Eq. (18) can be rearranged as

$$\begin{aligned}-\Delta\lambda_r/\lambda_r &\cong \lambda_r\{f_s\}^T[m_{ss}]^{-1}\left(([\Psi_s]^T)^{-1}[\Lambda_s][\Psi_s]^T - \lambda_r[I]\right)^{-1}\{f_s\}\\ &\le \lambda_r\{f_s\}^T[m_{ss}]^{-1}(\lambda_s - \lambda_r)^{-1}\{f_s\}\\ &= [\lambda_r/(\lambda_s - \lambda_r)]\{f_s\}^T[m_{ss}]^{-1}\{f_s\}\end{aligned} \tag{21}$$

or in detailed form

$$\begin{aligned}-\Delta\lambda_r/\lambda_r &= [\lambda_r/(\lambda_s - \lambda_r)]\left([m_{sp}]\{\phi_p\} + [m_{ss}]\{\phi_s\}\right)^T\\ &\times\left(\{\phi_s\} + [m_{ss}]^{-1}[m_{sp}]\{\phi_p\}\right)\\ &= [\lambda_r/(\lambda_s - \lambda_r)]\{\phi_s\}_{equi}[m_{ss}]\{\phi_s\}_{equi}\end{aligned} \tag{22}$$

where

$$\{\phi_s\}_{equi} = \{\phi_s\} + [m_{ss}]^{-1}[m_{sp}]\{\phi_p\} \tag{23}$$

Equation (23) can be rewritten as

$$[m_{ss}]\{\phi_s\}_{equi} = [m_{sp}]\{\phi_p\} + [m_{ss}]\{\phi_s\} = \{f_s\} \tag{24}$$

which shows that $\{\phi_s\}_{equi}$ represents the displacement of the secondary degrees of freedom under the inertia force $\{f_s\}$ with the primary set fixed.

In summary, the eigenvalue error is estimated as

$$-\Delta\lambda_r/\lambda_r \cong [\lambda_r/(\lambda_s - \lambda_r)]E_s \tag{25}$$

where

$$E_s = \{\phi_s\}_{equi}^T\{f_s\} = \{\phi_s\}_{equi}^T[m_{ss}]\{\phi_s\}_{equi} = \{f_s\}^T[m_{ss}]^{-1}\{f_s\} \tag{26}$$

The energy of the secondary set for $\{\phi_s\}_{equi}$ is denoted as E_s.

Equation (25) can be obtained directly from Eq. (18). The matrix inversion containing the eigenvalue is expanded using an infinite series[19]:

$$\begin{aligned}-\Delta\lambda_r/\lambda_r &= \lambda_r\{f_s\}^T[k_{ss}]^{-1}\{f_s\}\\ &+ \lambda_r^2\{f_s\}^T[k_{ss}]^{-1}[m_{ss}][k_{ss}]^{-1}\{f_s\} + \cdots\\ &= \lambda_r\{f_s\}^T\left([k_{ss}]^{-1}[m_{ss}]\right)[m_{ss}]^{-1}\{f_s\}\\ &+ \lambda_r^2\{f_s\}^T\left([k_{ss}]^{-1}[m_{ss}]\right)^2[m_{ss}]^{-1}\{f_s\} + \cdots\\ &\le (\lambda_r/\lambda_s)\{f_s\}^T[m_{ss}]^{-1}\{f_s\} + \left(\lambda_r^2|\ \lambda_s^2\right)\{f_s\}^T[m_{ss}]^{-1}\{f_s\} + \cdots\\ &\cong [\lambda_r/(\lambda_s - \lambda_r)]E_s, \qquad |\lambda_r/\lambda_s| < 1\end{aligned} \tag{27}$$

A poor selection of the primary degrees of freedom will push down λ_s closer to λ_r, and the denominator of Eq. (25) comes down to zero. In this case a drastic increase of frequency error or mode missing can occur. To investigate the relation with the ratio λ_r/λ_s, the energy of the secondary degrees of freedom E_s can be rewritten as

$$\begin{aligned}E_s &= \{\phi_s\}_{equi}^T[m_{ss}]\{\phi_s\}_{equi} = \{\phi_s\}_{equi}^T[k_{ss}][k_{ss}]^{-1}[m_{ss}]\{\phi_s\}_{equi}\\ &\le (1/\lambda_s)\{\phi_s\}_{equi}^T[k_{ss}]\{\phi_s\}_{equi} \le \lambda_r/\lambda_s\end{aligned} \tag{28}$$

The strain energy caused by $\{\phi_s\}_{equi}$ cannot be larger than the energy of the mode λ_r.

For higher modes a simple formulation can be derived. A refined expression for the secondary set is obtained as

$$\{\phi_s\}_{ref} \cong \{\phi_s\} + \lambda_r([k_{ss}] - \lambda_r[m_{ss}])^{-1}\{f_s\} \tag{29}$$

Using Eq. (29), Eq. (18) can be rewritten as

$$-\Delta\lambda_r/\lambda_r \cong (\{\phi_s\}_{ref} - \{\phi_s\})^T\{f_s\} \tag{30}$$

For the lowest modes good approximations correspond to $\{\phi_s\}_{ref} - \{\phi_s\} \cong \{0\}$.

In higher modes, however, considerable errors can be observed in the approximation, and a conservative estimation can be made as

$$\begin{aligned}|-\Delta\lambda_r/\lambda_r| &\cong \left|\{\phi_s\}_{ref}^T\{f_s\} - \{\phi_s\}^T\{f_s\}\right|\\ &\le \left|\{\phi_s\}_{ref}^T\{f_s\}\right| + \left|\{\phi_s\}^T\{f_s\}\right|\end{aligned} \tag{31}$$

When an approximate eigenmode is available, one gets

$$|-\Delta\lambda_r/\lambda_r| \le C\left|\{\phi_s\}_{app}^T\{f_s\}\right| \tag{32}$$

Weighting Factor for Row Sum

In the condensation the accuracy in the lowest modes is very important, whereas the Ritz vectors tend to contain higher modes. Therefore, a weighting procedure is recommended to put more emphasis on the lowest modes.

Equation (17) shows the relative error in eigenvalues, which indicates the imbalance of the strain energy. Hence, it can be expected that the energy distribution matrix has an error of the same order.

The weighting factor is defined as

$$\begin{aligned}w_r &= 1 - \lambda_r/\lambda_s\\ &\cong 1 - \lambda_r/\lambda_{L+1}, \qquad r = 1, \ldots, L\end{aligned} \tag{33}$$

where L denotes the number of accurate modes desired in the analysis.

The weighted row sum is calculated to determine the priority in the selection of the primary degrees of freedom:

$$Ew_i = \sum_{j=1}^{L} g_{ij}w_j = \sum_{j=1}^{L}\sum_{k=1}^{N}\phi_{ij}m_{ik}\phi_{kj}w_j \tag{34}$$

Summary of Solution Procedure

Figure 1 shows the flow diagram of the solution procedure, which starts with the calculation of the Ritz vectors. The weighted row sum of the energy distribution matrix is used to select the primary degrees of freedom. The column sum of the secondary set is calculated to estimate the accuracy in each mode.

Numerical Examples

L-Shaped Beam

For a numerical investigation of the method, flexural vibration of a uniform, L-shaped beam[8] is used as shown in Fig. 2. Six nodes and five elements are used for the finite element analysis. To simplify the problem, the motion will be planar, and hence the nodal displacement has 15 degrees of freedom. The exact eigenvalues in Table 1 were obtained through the use of MSC/NASTRAN.

To illustrate the energy variation of the degrees of freedom, the row sum of the energy distribution matrix was calculated using the exact modes. Because the system is small, all of the mode shapes were included, and no weighting was applied.

Table 1 Eigenvalues of L-shaped beam

Mode	Eigenvalue
1	8.93049×10^2
2	9.47189×10^3
3	1.43026×10^5
4	5.53071×10^5
5	1.25951×10^6
6	5.04416×10^6
7	8.14676×10^6

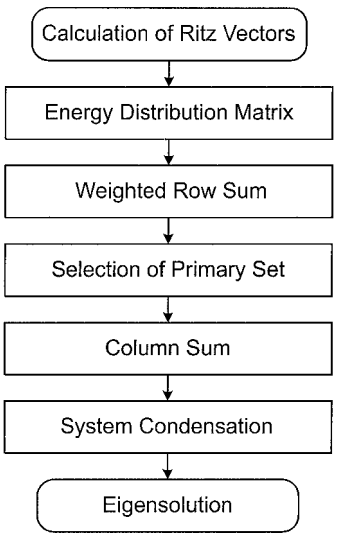

Fig. 1 Flow diagram of solution procedure.

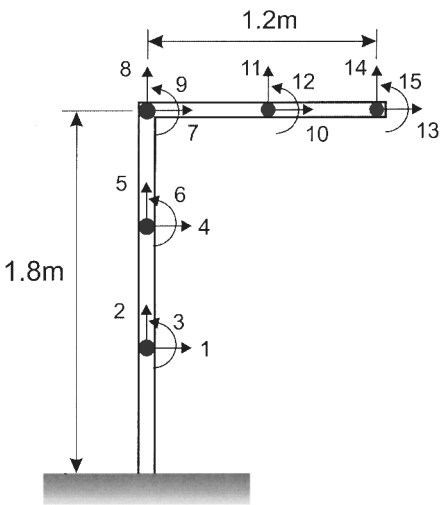

Fig. 2 L-shaped beam in flexural vibration.

Figures 3–5 show the energy variation of the degrees of freedom, which are divided into three groups dominating the energy distribution in the lower, intermediate, and higher modes, respectively. All of the plots reach unit value when the whole eigenmodes are involved.

In the lower modes the translational degrees of freedom carry large energy (Fig. 3), whereas the rotational ones gain significant energy in the intermediate modes (Fig. 4). Hence, the energy criterion has proven that the general rule of selecting translational degrees of freedom rather than rotational ones is appropriate only for the lowest modes.

Table 2 shows the sequential selection, which takes a degree of freedom with the largest energy in each mode. Because the vectors contain the higher modes, the corresponding degrees of freedom (for instance, 3 and 6) appear early.

Table 2 Sequential selection of primary set (beam)

Mode numbers	1	2	3	4	5	6	7	8	9	10
Exact modes	10	14	4	11	1	9	12	6	3	5
Ritz vectors	10	14	11	1	4	3	6	12	9	15

Table 3 Case study (beam)

Scheme	Primary degrees of freedom						
Exact modes	1	4	11	14	9	10	12
Ritz vectors	1	4	11	14	3	6	10
Weighted row sum	1	4	11	14	6	10	12
Shah	1	4	11	14	6	7	12
Matta	1	4	11	14	6	9	12

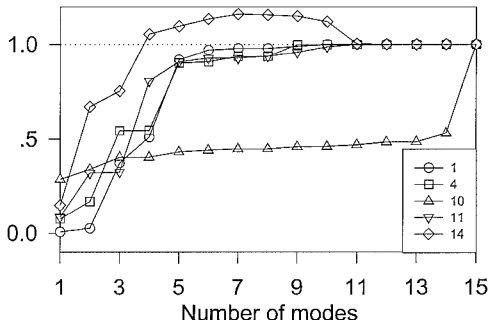

Fig. 3 Variation of energy, group 1 (beam).

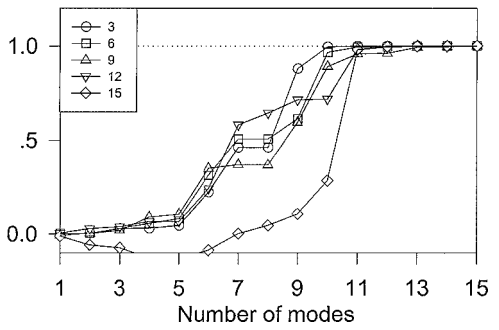

Fig. 4 Variation of energy, group 2 (beam).

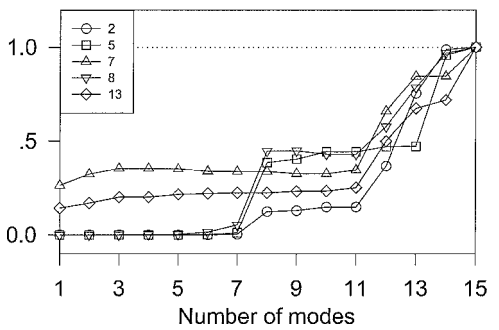

Fig. 5 Variation of energy, group 3 (beam).

To compare several selection schemes, the primary set is limited to seven degrees of freedom, as shown in Table 3. From the degrees of freedom 7 and 10, the energy method (upper three) takes 10, whereas Shah chooses 7.

Static condensations with the primary sets give relative errors in eigenvalue, as shown in Table 4. If proper selections are made, accurate solutions are obtained for up to the fifth mode. The results of the weighted row sum and Shah indicate that the degrees of freedom 7 and 10 have the same dynamic characteristics.

Table 5 shows the column sum of energy of the secondary set. The large magnitude does not mean a big error in the first mode, for

which Eqs. (17) and (25) should be used. The criterion of the simple column sum is applicable for the intermediate modes in which the mode shapes have a noticeable difference. The sudden increase of energy in mode 6 implies the transition point.

To get accurate solutions for up to the seventh mode, the primary set was expanded as shown in Table 6. The weighted row sum calculated for the lowest seven modes is used to decide the priority in selection. A total of 10 degrees of freedom with the highest row sums are illustrated in order of magnitude.

The expansion of the primary set leads to an energy decrease of the secondary set, and the error reduction of the sixth and seventh eigenvalues, as shown in Table 7. The inclusion of the degree of freedom 7 does not improve the solution accuracy, which can be expected from the negligible change of the column sum of energy.

The numerical results show that the row sum should be larger than 0.35–0.40 to get an accurate solution in the typical example problem. The column sum of the secondary set should be less than 0.10.

Table 4 Error in eigenvalues, % (beam)

Mode	Exact	Ritz	Weighted row sum	Shah	Matta
1	0.00	0.00	0.00	0.00	0.46
2	0.05	0.07	0.05	0.05	1.10
3	0.32	0.22	0.19	0.21	12.15
4	0.66	2.43	1.16	1.16	1.51
5	0.91	0.28	0.71	0.82	32.33
6	24.47	24.28	33.87	33.97	12.78
7	162.68	214.03	26.41	26.75	67.36

Table 5 Column sum of secondary set (beam)

Mode	Exact	Ritz	Weighted row sum	Shah	Matta
1	0.40	0.39	0.39	0.41	0.68
2	0.04	0.07	0.05	0.05	0.09
3	0.10	0.07	0.09	0.12	0.14
4	0.02	0.05	0.02	0.02	0.05
5	0.05	0.02	0.04	0.07	0.05
6	0.46	0.46	0.47	0.49	0.23
7	0.57	0.50	0.40	0.41	0.39

Table 6 Expansion of primary set (beam)

Order	Degree of freedom	Weighted row sum	Row sum
1	14	1.12	1.16
2	1	0.92	0.92
3	11	0.90	0.90
4	4	0.89	0.94
5	10	0.44	0.45
6	12	0.37	0.58
7	6	0.35	0.51
8	7	0.34	0.34
9	3	0.30	0.46
10	9	0.29	0.37

Table 7 Reduction of error through set expansion (beam)

Order	Degree of freedom	Column sum		Error in eigenvalues, %	
		Mode 6	Mode 7	Mode 6	Mode 7
1–7	——	0.47	0.40	33.9	26.4
8	7	0.48	0.40	33.8	26.3
9	3	0.30	0.17	15.8	9.4
10	9	0.06	0.15	1.7	5.5

Helicopter Tail-Boom

The tail-boom structure of the Cobra helicopter[18] is used as an example of moderate size. The finite element model has 28 nodes, 48 beam elements, and 8 concentrated masses. Figure 6 shows the geometrical layout of the system.

With four nodes at the support completely fixed, the structure has 144 unconstrained degrees of freedom. Table 8 shows the first seven eigenvalues. The average of the consistent and lumped masses was used for better accuracy.

The exact eigenmodes were used to get the energy variation in Fig. 7. Four nodes at a cross-sectional frame show the same vibration characteristics. Hence, nodes 12–17 were used to represent the frames. The longitudinal translations (17x, 14x) gain much energy in the seventh mode and hence should be included in the primary set to get an accurate approximation.

Table 9 shows the primary degrees of freedom obtained through the sequential selection. It can easily be expected that the nodes

Table 8 Eigenvalues of tail-boom

Mode	Eigenvalue	Characteristics
1	1.72244×10^3	First bending (about y axis)
2	1.79624×10^3	First bending (about z axis)
3	8.76679×10^3	First twisting (about x axis)
4	1.48218×10^4	Second bending (about y axis)
5	1.51423×10^4	Second bending (about z axis)
6	6.28409×10^4	Second twisting (about x axis)
7	1.93407×10^5	First axial (along x axis)

Table 9 Sequential selection of primary set (tail-boom)

Modes	1	2	3	4	5	6	7	8
Exact modes	17y	17z	14y	14z	13y	13z	17x	12z
Ritz vectors	17y	17z	14z	14y	17x	14x	12z	12y

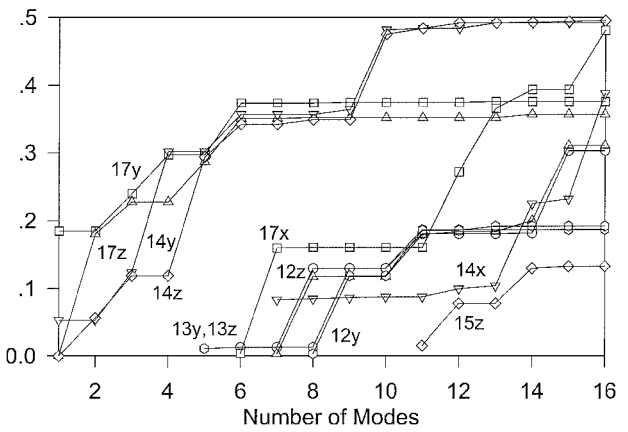

Fig. 7 Variation of energy (tail-boom).

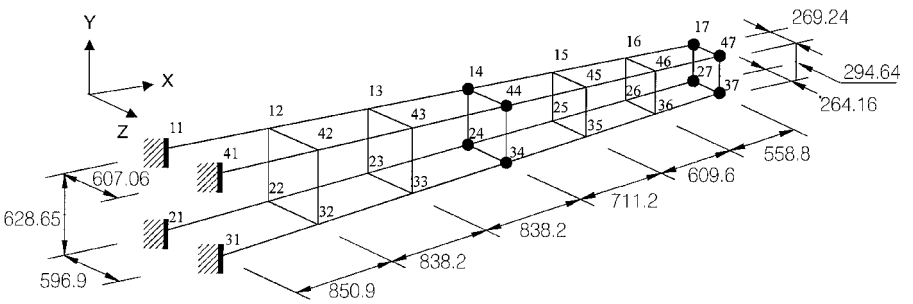

Fig. 6 Helicopter tail-boom.

Table 10 Case study (tail-boom)

	Primary set		
Variable	1	2	3
Degrees of freedom	14y, 14z 17y, 17z	14y, 14z 17y, 17z 13y, 13z	14y, 14z 17y, 17z 14x, 17x
Dimension	16	24	24

Table 11 Error in eigenvalues, % (tail-boom)

	Primary set		
Mode	1	2	3
1	0.01	0.01	0.01
2	0.01	0.01	0.01
3	0.05	0.03	0.05
4	0.17	0.05	0.14
5	0.18	0.05	0.14
6	0.23	0.04	0.23
7	197.41	125.56	0.03

Table 12 Column sum of secondary set (tail-boom)

	Primary set		
Mode	1	2	3
1	0.05	0.05	0.05
2	0.05	0.05	0.05
3	0.07	0.05	0.07
4	0.06	0.03	0.05
5	0.06	0.03	0.05
6	0.03	0.02	0.03
7	0.99	0.99	0.06

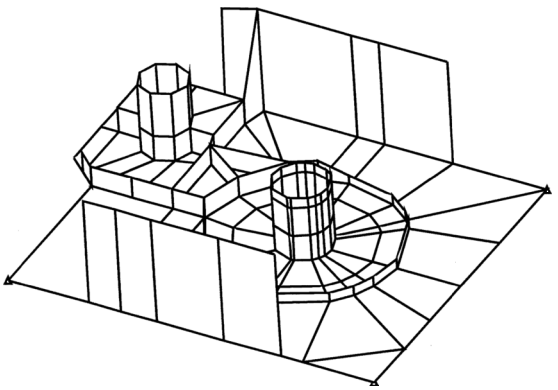

Fig. 8 Aluminum casting of disk drive.

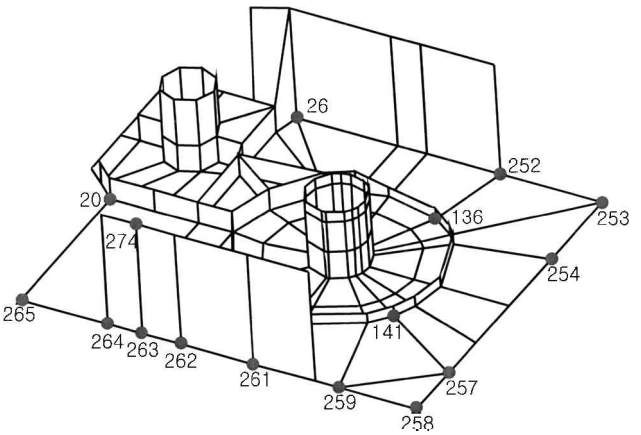

Fig. 9 Primary nodes (disk drive, energy method).

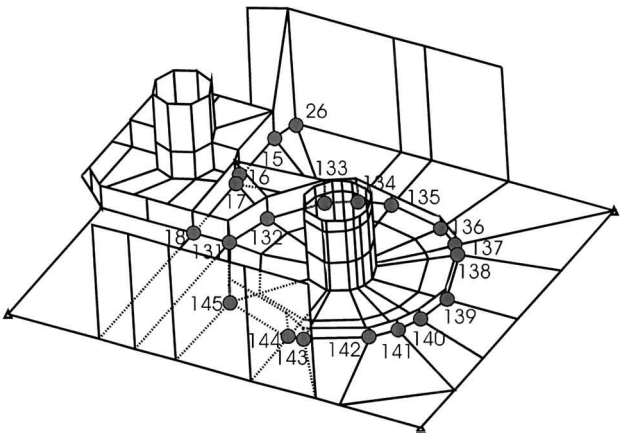

Fig. 10 Primary nodes (disk drive, intuition).

Table 13 Eigenvalues of disk drive

		Condensation	
Mode	Exact	Energy method	Intuition
1	1.18554×10^2	1.18554×10^2	(Missing)
2	1.19824×10^2	1.19824×10^2	(Missing)
3	2.32684×10^2	2.32684×10^2	2.32417×10^2
4	3.32132×10^2	3.32132×10^2	(Missing)
5	5.27411×10^2	5.27411×10^2	5.12613×10^2
6	8.44785×10^2	8.44786×10^2	8.19986×10^2
7	5.69953×10^6	5.78984×10^6	8.95220×10^6
	——	(1.58%)	(57.07%)

carrying the concentrated masses, 14 and 17, have large energy and should be included in the analysis. The Ritz vectors show the same tendency as before. The early appearance of the longitudinal translations, 17x and 14x, is noticed. The vectors contain considerable portions of the higher modes and hence overestimate the eigenvalues.

To emphasize the importance of the longitudinal displacements for the seventh mode, three primary sets in Table 10 were considered. The third one contains 14x and 17x. All of the four nodes at each cross-sectional frame need to be included, and hence the dimension of the primary set increases.

Table 11 shows the corresponding relative error in the first seven eigenvalues. The primary sets 1 and 2 give excellent results for modes up to six but, in fact, miss the seventh mode. The column sum of the secondary degrees of freedom is illustrated in Table 12. The energy is related with the solution accuracy in the condensation.

Aluminum Casting of Disk Drive

As a case study on a large problem, an aluminum casting of disk drive[23] was considered. A total of 144 beams, 8 trusses, and 160 plates were used for the finite element modeling, as shown in Fig. 8. To constrain the rigid-body motions, the four corners are supported with soft springs. The structural analysis has 1053 unconstrained degrees of freedom.

MSC/NASTRAN was used to calculate the eigenmodes. Using the exact mode shapes, the energy distribution matrix was calculated with DMAP statements, and the energies were added over the seven modes with weighting. Figure 9 shows the nodes associated with 20 primary degrees of freedom.

For comparison, another primary set was selected arbitrarily on engineering intuition. Because the first mode of elastic vibration indicates twisting of the floor, the nodes connected to the vertical elements are included, as shown in Fig. 10.

Table 13 shows the exact eigenvalues and those obtained through the static condensation. The first six modes represent the rigid motions, and mode 7 is the first elastic vibration of floor twisting. The energy method gives a good solution, whereas three rigid-body motions are missing in the intuitive selection.

Table 14 indicates that the column sum of the secondary set should be less than 0.4–0.5. If the value is very close to unity, so is the ratio λ_r / λ_s. Then the solution diverges, and even the lowest modes are missing. The row sum of the primary set ranges from 0.06 to 0.25.

Table 14 Column sum of secondary set (disk drive)

Mode	Energy method	Intuitions
1	0.904	0.999
2	0.919	0.999
3	0.705	0.740
4	0.884	0.999
5	0.545	0.840
6	0.500	0.899
7	0.460	0.948

Conclusions

The energy method has proved to work well for the selection of the primary degrees of freedom in condensation of eigenproblems. Numerical examples show that the criterion can provide a guideline for which and how many degrees of freedom should be included in the analysis. The method is based on the energy distribution over the degrees of freedom in the eigenmodes of structural systems.

The primary set takes the degrees of freedom making dominant contributions to the lowest modes. The usual practice of selecting the translational degrees of freedom is appropriate only for the lowest modes. The rotational ones carry significant energy in the intermediate and higher modes.

For the energy estimation Ritz vectors are calculated using the stiffness and mass matrices. Because the vectors contain intermediate and higher eigenmodes, the corresponding degrees of freedom are introduced early into the primary set. To put more emphasis on the lowest modes, a weighting procedure is recommended. The method is based on the physical meaning of the energy distribution and hence seems to be more rational than other selection schemes.

Further study is needed to elaborate the energy criterion. It is believed that the method can also be used in structural optimization, system identification, optimal sensor placement, and other cases where a reduced subsystem should represent the dynamic characteristics of the whole system.

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